Threshold Effects in Average Cross Sections According to *R*-Matrix Theory*

W. E. MEYERHOF
Department of Physics, Stanford University, Stanford, California
(Received 13 August 1962)

This paper attempts to place on a firm basis certain expressions for effects in elastic and total cross sections caused by—and in the neighborhood of—a reaction threshold. Explicit expressions are derived for the analytic behavior of the collision matrix near a reaction threshold. These expressions are based on the *R*-matrix theory of nuclear reactions and extend slightly work by Wigner and by Breit on threshold effects. The expressions are quite general, allowing for the presence of compound resonances. Both the channel matrix and the level matrix formulations of *R*-matrix theory are used. The former turns out to be convenient for formulating the general expressions of the collision matrix and the cross sections. The latter is more convenient for performing energy averages.

It is shown that certain formulas for energy-averaged total and elastic cross sections, which were made plausible in a previous paper by the author, follow from the above-mentioned general expressions by performing suitable energy averages. Consequences of the random-sign approximation of the "value quantities" γ_λ and of a partial breakdown of this assumption are examined and related to the assumptions of the optical model. It is shown that under the assumption of random signs, the total cross section should show no threshold effects, whereas if this assumption is relaxed threshold effects appear. Hence it is, in principle, possible to decide by experiment which situation obtains. Finally, cross-section threshold effects under the single-level approximation are given; with a slight generalization of the phase shift, these are identical to expressions derived by Baz and by Newton.

I. INTRODUCTION

IN a previous paper[1] we had examined threshold effects in elastic scattering under the assumption that the energy-averaged diagonal component of the collision matrix is equal to the optical-model collision matrix. This is, of course, the basic assumption of the optical model[2] and implies that the energy-averaged diagonal component of the collision matrix (and hence the energy-averaged total cross section) will not show any particular discontinuities at reaction thresholds. In I we made it plausible that threshold effects in energy-averaged cross sections, usually called compound competition effects, can be deduced by properly energy averaging the Wigner cusps,[3] which are expected to occur in cross sections near reaction thresholds. Therefore, it appeared that there is no essential difference between compound competition effects and Wigner cusps. It should be noted, though, that compound competition effects can be derived solely from the unitarity of the collision matrix whereas Wigner cusps require in addition an analytic continuation of the collision matrix across reaction thresholds. Extending slightly Wigner's work[3] and an investigation of Breit,[4] we shall show that on the basis of *R*-matrix theory such analytic continuation always exists, even in the presence of many narrow and possibly overlapping resonances. This means that in derivations of threshold effects there is no need to require that certain phase shifts vary slowly with energy in the cusp region, as is usually assumed,[5–7] explicitly or implicitly.

By taking energy averages of the general cross-section expressions, subject to the implicit assumption of random signs[8] of the value quantities $\gamma_{\lambda c}$ introduced by Wigner and Eisenbud,[9] we shall show that the optical-model assumptions and compound competition effects in cross sections are obtained near threshold.[1]

This derivation leans heavily on work of Thomas.[8] If the random sign assumption is relaxed for distant resonances,[10,11] which represents a direct interaction mechanism, threshold effects are found to occur in the total cross section contrary to the usual optical-model assumption. Finally, if the one-level approximation for the R matrix is made,[12] we obtain formulas for the threshold effects in cross sections similar to those of others,[5–7] with the exception that the relevant phase shift is allowed to vary with the energy of the incident particle.

As in I, we restrict ourselves to the situation where the reaction threshold is the one of lowest energy. This does not represent any fundamental limitation, but simplifies the presentations. From the same point of view, we shall discuss only the case of one entering channel (called s) and one channel (called t) which will be emergent above threshold. All other channels of the problem will then be negative energy channels and can be eliminated by use of the reduced R matrix of Wigner and Teichmann. The more useful case of two entering channels can be treated by a similar method and yields

* Supported in part by the U. S. Army Research Office (Durham) and the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

[1] W. E. Meyerhof, Phys. Rev. **128**, 2312 (1962). This paper will be referred to as I; references to equations in this paper will be preceded by I, e.g., I Eq. (19).

[2] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

[3] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).

[4] G. Breit, Phys. Rev. **107**, 1612 (1957) and in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part. 1, p. 274 ff.

[5] A. I. Baz, Soviet Phys.—JETP **6**, 709 (1959).

[6] R. G. Newton, Phys. Rev. **114**, 1611 (1959).

[7] L. Fonda, Nuovo Cimento **20**, 116 (1961).

[8] R. G. Thomas, Phys. Rev. **97**, 224 (1955).

[9] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[10] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958). Since this review article is rather extensive we shall, for the convenience of the reader, give references to particular equations or sections of the article.

[11] Reference 10, Sec. XI, 6.

[12] Reference 10, Sec. XII, 1.

the somewhat more complicated cross-section expressions given in I for target spin zero and bombarding-particle spin one-half. For simplification, also, we shall discuss only integrated cross sections. Expressions for the differential cross sections follow immediately from reference 5 or the treatment in I, once the threshold effect in the collision matrix is known.[13]

II. PRELIMINARIES

A. Definitions and Posing of Problem

We shall follow, as much as possible, the notations and definitions of Lane and Thomas.[10] Whenever the clarity of presentation is not impaired we do not repeat any derivation given there.

The running index for channel quantities will be c or c' and for level quantities λ, μ, or ν. All cross sections will be expressed in units of $\pi\lambda\!\!\!^{-}{}_s^2 g_s$, where $\lambda\!\!\!^{-}{}_s$ is the reduced De Broglie wavelength in the entering channel and g_s is a statistical factor.[14] For the case of spin-zero target and bombarding particle, $g_s=2l_s+1$. As in I we assume that the threshold effect occurs only in one entering partial wave (called l_s here). Hence it will be sufficient for us to consider only the partial wave cross sections in terms of the components of the collision matrix.[15]

Integrated elastic cross section:

$$\sigma_{ss}=|1-U_{ss}|^2, \tag{1}$$

Reaction cross section ($s\rightarrow t$):

$$\sigma_{st}=|U_{st}|^2=1-|U_{ss}|^2, \tag{2}$$

Total cross section (for s):

$$\sigma_s=2(1-\mathrm{Re}U_{ss}). \tag{3}$$

We omit any superscript designation l_s on the partial cross sections and the collision matrix components, since "channel s" implies here that the orbital angular momentum has the value l_s.

For calculation of the threshold effect we shall find it convenient to introduce the modified collision matrix $\mathbf{W}$, which is related[16] to $\mathbf{U}$ by

$$\mathbf{U}=\mathbf{\Omega W \Omega}. \tag{4}$$

$\mathbf{\Omega}$ is a diagonal matrix with components[17]

$$\Omega_c=e^{i\varphi_c}, \quad \varphi_c=\omega_c-\phi_c, \tag{5}$$

where ω_c is a Coulomb phase shift[18] and ϕ_c the so-called hard-sphere phase shift. There are several equivalent relationships between $\mathbf{W}$ and the R matrix; the most convenient for us is[16,19]

$$\mathbf{W}=\mathbf{1}-2i\mathbf{Pq}+2i\mathbf{P}^{1/2}\mathbf{q}(\mathbf{q}-\mathbf{R})^{-1}\mathbf{P}^{1/2}\mathbf{q}. \tag{6}$$

$\mathbf{q}$ is a diagonal matrix which is the inverse of the effective logarithmic derivative[16] $\mathbf{L}^0$:

$$\mathbf{q}=(\mathbf{L}^0)^{-1}, \quad \mathbf{L}^0=\mathbf{S}^0+i\mathbf{P}, \quad \mathbf{S}^0=\mathbf{S}-\mathbf{B}, \tag{7}$$

where all matrices are diagonal. S_c is the shift factor,[17] P_c the penetration factor,[17] and B_c is an energy-independent boundary value quantity,[20] which often allows the effective shift function $\mathbf{S}^0$ to be set equal to zero. S_c, P_c, and B_c are real by definition.

As mentioned in footnote 16 of I, it follows from Eq. (2) that for the entering (c.m.) energy E_s above the threshold energy E_{thr}

$$U_{ss}\cong e^{2i\delta_s}(1-\tfrac{1}{2}\sigma_{st}), \quad E_s>E_{\mathrm{thr}}, \tag{8}$$

where the approximate sign implies that $\sigma_{st}\ll 1$. The questions which were not considered in I, and which we shall answer below, are:

(1) What exactly is the phase shift δ_s in Eq. (8)?

(2) How is expression (8) to be continued analytically for $E_s<E_{\mathrm{thr}}$?

Once these questions have been answered, simple substitutions in Eqs. (1) and (3) give the desired threshold effects in the cross sections.

B. Logarithmic Derivative

Since the logarithmic derivative[17] $L_c=S_c+iP_c$, plays an important role in the subsequent development, we wish to give its limiting values[21] in the absence of Coulomb effects for low and high channel energies E_c. These values are expressed in terms of the quantity ρ_c (here defined to be valid for positive and negative E_c, denoted by superscripts + and − below)

$$\rho_c=k_c a_c, \quad \text{where} \quad k_c=(2M_c/\hbar^2)^{1/2}|E_c|^{1/2} \tag{9}$$

and a_c is the channel radius. M_c is the reduced mass in channel c.

The s-wave case is of particular importance and requires special consideration.

$l_c=0$, all ρ_c:

$$S_c^+=0, \quad P_c^+=\rho_c, \quad S_c^-=-\rho_c; \tag{10}$$

[13] Certain aspects in Secs. II to V of the present paper have been discussed by R. H. Capps and W. G. Holladay, Phys. Rev. **99**, 931 (1955), Appendix B; R. K. Adair, Phys. Rev. **111**, 632 (1958); A. N. Baz and L. B. Okun, Soviet Phys.—JETP **8**, 526 (1959); J. Sucher, G. A. Snow, and T. B. Day, Phys. Rev. **122**, 1645 (1961).

[14] Reference 10, Sec. VIII, 3.

[15] Reference 10, Chap. VI.

[16] Reference 10, Sec. VII, 1.

[17] Reference 10, Sec. III, 4.

[18] Reference 10, Sec. III, 2.

[19] This expression corrects two misprints in reference 10, Eq. VII (1.6b). It is completely equivalent to similar equations in references 3 and 4, as can be seen by noting that $\mathbf{1}-2i\mathbf{Pq}=\mathbf{q}/\mathbf{q}^*$ and by letting the quantities $\mathbf{e}^{-1}$ defined in these references be proportional to $\mathbf{\Omega P}^{1/2}\mathbf{q}$.

[20] Reference 10, Sec. V, 2.

[21] For the evaluation of Eqs. (10) and (12) we used expressions for S_c and P_c given by J. E. Monahan, L. C. Biedenharn, and J. P. Schiffer, Argonne National Laboratory Report ANL-5846, 1958 (unpublished).

$l_c \neq 0, \quad \rho_c \ll 1$:

$$S_c^+ \cong -l_c + \rho_c^2/(2l_c-1), \tag{11a}$$

$$P_c^+ \cong 2^{l_c}[l_c!/(2l_c!)^2]\rho_c^{2l_c+1}, \tag{11b}$$

$$S_c^- \cong -l_c - \rho_c^2/(2l_c-1) - (-2)^{l_c}[l_c!/(2l_c!)^2]\rho_c^{2l_c+1}; \tag{11c}$$

$l_c \neq 0, \quad \rho_c \gg 1$:

$$S_c^+ \cong \tfrac{1}{2} l_c(l_c+1)/\rho_c^2, \tag{12a}$$

$$P_c^+ \cong \rho_c, \tag{12b}$$

$$S_c^- \cong -\rho_c. \tag{12c}$$

In all cases $P_c^- = 0$. In expression (11c) we have indicated that the lowest power of ρ_c is always ρ_c^2, but the lowest odd power (which gives rise to cusps in derivatives of the cross sections[4]) is $\rho_c^{2l_c+1}$.

C. Reduced *R* Matrix for Two Channels

Since the threshold under consideration is the one of lowest energy, all channels besides s and t will have negative energy near $E_s = E_{\mathrm{thr}}$ and the reduced R matrix is real.[22] The relation between the reduced R matrix and the collision matrix is still given by Eqs. (4) and (6) and in addition $\mathbf{W}$ is unitary since the reduced R matrix is real.[23]

Although for the general demonstration of analytic continuity of Eq. (8) we do not require that the reduced R matrix be real, it turns out that a real matrix is more convenient later on and we need the form of this matrix[23] for later work:

$$\mathfrak{R}_{sc} = \textstyle\sum_{\lambda\mu} \gamma_{\lambda s}\gamma_{\mu c}(\mathbf{A})_{\lambda\mu}, \quad c = s \text{ or } t, \tag{13}$$

where, as long as E_s is below the energy of the second lowest reaction threshold,

$$(\mathbf{A}^{-1})_{\lambda\mu} = (E_\lambda - E_s)\delta_{\lambda\mu} - \sum_{c' \neq s,t} S_{c'}^{0-}\gamma_{\lambda c'}\gamma_{\mu c'}. \tag{14}$$

From Eq. (7) we see that $S_{c'}^{0-}$ could be set equal to zero at any energy, making the reduced R matrix equal to the ordinary R matrix.

A better approximation for an extended energy range is to expand $\mathbf{A}$ about its diagonal components in increasing powers of its off diagonal components. Two approximations to $\mathfrak{R}_{sc}$ are then available. Near any given energy E_s one can choose $B_{c'}$ very close to $S_{c'}^-$ so that all the off diagonal elements, which are proportional to $S_{c'}^{0-} (= S_{c'}^- - B_{c'})$, become negligibly small compared to the diagonal components. In this case[22]

$$\mathfrak{R}_{sc} \cong \textstyle\sum_\lambda \gamma_{\lambda s}\gamma_{\lambda c}/(E_\lambda + \Delta_{\lambda\lambda} - E_s), \tag{15a}$$

where

$$\Delta_{\lambda\lambda} = -\sum_{c' \neq s,t} S_{c'}^{0-}\gamma_{\lambda c'}^2$$

is the level shift of the negative-energy channels. If one assumes that it is valid to expand $S_{c'}^{0-}$ linearly about the energy E_s, Eq. (15a) can be brought into the form[22]

$$\mathfrak{R}_{sc} \cong \textstyle\sum_\lambda \gamma_{\lambda s}^0\gamma_{\lambda c}^0/(E_\lambda - E_s), \tag{15b}$$

where

$$\gamma_{\lambda c}^0 = \gamma_{\lambda c}\Big[1 + \sum_{c' \neq s,t} (dS_{c'}^{0-}/dE_s)\gamma_{\lambda c'}^2\Big]^{-1/2}. \tag{16}$$

A second approximation to Eq. (13) was obtained by Thomas[8] under the assumption of random signs of the quantities $\gamma_{\lambda c}$. In this case it can be argued[8] that Eq. (15a) is a valid approximation to $\mathfrak{R}_{sc}$ in any energy region (below the second lowest reaction threshold), as long as the partial level widths for the channels $c' \neq s, t$ are less than the spacings of the levels. The important point is, though, that there are no restrictions on the partial width to spacing ratios for the channels s and t.

III. CHANNEL MATRIX FORMULATION OF THRESHOLD EFFECTS

Analytic Continuation of U_{ss} Below E_{thr}

With the use of the reduced R matrix, $\mathbf{W}$ [Eq. (6)] becomes 2×2 matrix; the only minor problem is the inversion of the matrix $\mathbf{q} - \mathfrak{R}$ which yields

$$(\mathbf{q} - \mathfrak{R})^{-1} = \begin{pmatrix} (q_t - \mathfrak{R}_{tt})/d & \mathfrak{R}_{st}/d \\ \mathfrak{R}_{st}/d & (q_s - \mathfrak{R}_{ss})/d \end{pmatrix}, \tag{17}$$

where

$$d = (q_s - \mathfrak{R}_{ss})(q_t - \mathfrak{R}_{tt}) - \mathfrak{R}_{st}^2, \tag{18}$$

since $\mathfrak{R}$ is symmetric.[23] From this we get expressions similar to those given[24] in reference 10,

$$W_{ss} = 1 - 2iP_sq_s + 2iP_sq_s^2(q_t - \mathfrak{R}_{tt})/d, \tag{19}$$

$$W_{st} = 2i(P_sP_t)^{1/2}q_sq_t\mathfrak{R}_{st}/d. \tag{20}$$

By setting $B_t = l_t$, we see from expressions (10), (11), and (7) that $q_t \to \infty$ as $E_s \to E_{\mathrm{thr}}$, i.e. $\rho_t \to 0$. Hence we can expand Eqs. (19) and (20) in a manner proposed by Wigner[3] and by Breit,[4] keeping only the lowest negative power of q_t (or lowest positive power of L_t^0). After a little algebra we find

$$W_{ss} = 1 + \frac{2iP_sq_s\mathfrak{R}_{ss}}{q_s - \mathfrak{R}_{ss}} + \frac{2iP_sq_s^2\mathfrak{R}_{st}^2}{(q_s - \mathfrak{R}_{ss})^2} \times \left(\frac{1}{q_t - \mathfrak{R}_{tt}} + \frac{\mathfrak{R}_{st}^2}{(q_s - \mathfrak{R}_{ss})(q_t - \mathfrak{R}_{tt})^2} + \cdots\right),$$

$$W_{st} = 2i(P_sP_t)^{1/2}\frac{q_sq_t\mathfrak{R}_{st}}{(q_s - \mathfrak{R}_{ss})(q_t - \mathfrak{R}_{tt})} \times \left(1 + \frac{\mathfrak{R}_{st}^2}{(q_s - \mathfrak{R}_{ss})(q_t - \mathfrak{R}_{tt})} + \cdots\right).$$

---

[22] Reference 10, Sec. X, 2.

[23] Reference 10, Sec. X, 1.

[24] Reference 10, Sec. X, 3.

In each case we have kept the next higher order terms to show what will be neglected. If we consider an energy region where $q_t \gg \mathfrak{R}_{tt}, \mathfrak{R}_{st}$, we finally obtain

$$W_{ss} \cong 1+\frac{2iP_s\mathfrak{R}_{ss}}{1-L_s^0\mathfrak{R}_{ss}}+\frac{2iL_t^0P_s\mathfrak{R}_{st}^2}{(1-L_s^0\mathfrak{R}_{ss})^2} \quad (21a)$$

$$=\frac{1-L_s^{0*}\mathfrak{R}_{ss}}{1-L_s^0\mathfrak{R}_{ss}}\left(1+\frac{2iL_t^0P_s\mathfrak{R}_{st}^2}{|1-L_s^0\mathfrak{R}_{ss}|^2}\right), \quad (21b)$$

$$W_{st} \cong 2i(P_sP_t)^{1/2}q_s\mathfrak{R}_{st}/(q_s-\mathfrak{R}_{ss}),$$

$$|W_{st}|^2 \cong 4P_sP_t\mathfrak{R}_{st}^2/|1-L_s^0\mathfrak{R}_{ss}|^2, \quad (22)$$

where we have made use of expressions (7).

Before proceeding we must show that the same expressions are obtained even if $\mathfrak{R}_{ss}$, $\mathfrak{R}_{st}$, and $\mathfrak{R}_{tt}$ become infinitely large, as they would if E_s approaches any resonance energy E_λ [see Eq. (15b)]. Let us assume, then, that E_s is very close to one particular resonance energy E_λ so that

$$\mathfrak{R}_{sc} \cong \gamma_{\lambda s}^0\gamma_{\lambda c}^0/(E_\lambda-E_s). \quad (23)$$

Substitution in Eq. (18) gives

$$d \cong q_sq_t-[q_s(\gamma_{\lambda t}^0)^2+q_t(\gamma_{\lambda s}^0)^2]/(E_\lambda-\gamma_s) \quad (24)$$

and from (19) we obtain

$$W_{ss} \cong 1+\frac{2iP_sq_sq_t(\gamma_{\lambda s}^0)^2}{q_sq_t(E_\lambda-E_s)-q_s(\gamma_{\lambda t}^0)^2-q_t(\gamma_{\lambda s}^0)^2}.$$

Expanding for large q_t and using Eq. (7)

$$W_{ss} \cong 1+\frac{2iP_s(\gamma_{\lambda s}^0)^2}{E_\lambda-E_s-L_s^0(\gamma_{\lambda s}^0)^2} +\frac{2iL_t^0P_s(\gamma_{\lambda s}^0)^2(\gamma_{\lambda t}^0)^2}{[E_\lambda-E_s-L_s^0(\gamma_{\lambda s}^0)^2]^2}, \quad (25)$$

which is identical to Eq. (21a) under the assumption (23). In a similar way, it is easily shown that with the assumption (23)

$$|W_{st}|^2 \cong 4P_sP_t(\gamma_{\lambda s}^0)^2(\gamma_{\lambda t}^0)^2/|E_\lambda-E_s-L_s^0(\gamma_{\lambda s}^0)^2|^2 \quad (26)$$

to lowest order in P_t or L_t^0. This is identical to Eq. (22) under the condition (23).

Hence, independently how close the energy E_s is to any resonance energy E_λ, the same relationship is obtained between W_{ss} and $|W_{st}|^2$, which from Eqs. (21b) and (22) [or Eqs. (25) and (26)] is

$$W_{ss} \cong \frac{1-L_s^{0*}\mathfrak{R}_{ss}}{1-L_s^0\mathfrak{R}_{ss}}\left[1+\left(\frac{iL_t^0}{P_t^+}\right)\frac{|W_{st}|^2}{2}\right]. \quad (27)$$

The superscript + on P_t^+ emphasizes that $|W_{st}|^2$ exists only for positive energies E_t. Nevertheless, if we express L_t^0 and P_t^+ as functions of the absolute quantity ρ_t defined by Eq. (9), the relationship (27) is valid both above and below the threshold energy for channel t. Hence it provides the desired analytic continuation for W_{ss} below threshold. As implied already by Wigner,[3] this continuation hinges on the analytic properties of L_t^0, which can be recognized in Eqs. (10) and (11). All that is needed, is to substitute in L_t^0, $\rho_t \rightarrow i\rho_t$ in going from $E_s > E_{\text{thr}}$ to $E_s < E_{\text{thr}}$.[25] The approximate sign in Eq. (27) implies only that all quantities have been expanded to lowest order in L_t^0 or P_t^+.

By defining a phase angle δ_s through

$$(1-L_s^{0*}\mathfrak{R}_{ss})/(1-L_s^0\mathfrak{R}_{ss})=e^{2i(\delta_s-\varphi_s)}, \quad (28)$$

we recognize that Eq. (27) is practically equivalent to Eq. (8), recalling Eq. (5) and $\sigma_{st}=|W_{st}|^2$, as can be seen from Eq. (2). Equation (27) therefore will give the analytically continuable form of U_{ss} in the neighborhood of E_{thr}, which we set out to find. From Eqs. (10) and (11a) we find, setting $B_t=l_t$ [see Eq. (7)]

for $l_t=0$:

$$U_{ss}=e^{2i\delta_s}\left(1-\frac{1}{2}\sigma_{st}\left\{\begin{matrix}1\\ i\end{matrix}\right.\right), \quad (29a)$$

for $l_t \neq 0$:

$$U_{ss}=e^{2i\delta_s}\left(1+i\frac{\rho_t^{1-2l_t}\sigma_{st}}{2^{l_t+1}[l_t!/(2l_t!)^2](2l_t-1)} \times\left\{\begin{matrix}1\\ -1\end{matrix}\right. -\frac{1}{2}\sigma_{st}\left\{\begin{matrix}1\\ i(-1)^{l_t}\end{matrix}\right.\right). \quad (29b)$$

Following Newton[6] the upper line after each curly bracket refers to $E_s > E_{\text{thr}}$, the lower line to $E_s < E_{\text{thr}}$; this convention will be used throughout the rest of this paper. We see from Eqs. (28) and (29) that if δ_s is defined as that part of the phase of U_{ss} which is completely independent of ρ_t, Eq. (8) is not quite correct. On the other hand, the second term in the bracket of Eq. (29b) does not give rise to any discontinuity at E_{thr} in any cross section or its derivatives[4] since Eqs. (11b) and (22) show that this term is proportional to ρ_t^2 independent of l_t. Hence one can absorb this term in the phase δ_s, if one wishes, and this was done in I Eqs. (21) to (23) and I Eq. (32), the last of which was taken from the work of Baz.[5]

In the rest of this paper we shall use the definition (28) of the phase δ_s, thus eliminating all threshold effects from it.

IV. LEVEL MATRIX FORMULATION OF THRESHOLD EFFECTS

The channel matrix approach of the previous section gives a very clear derivation for the form of U_{ss} near E_{thr}, but expression (27) does not lend itself readily to

[25] This is sufficient if there are no Coulomb effects in the outgoing channel. For the case which includes Coulomb effects see reference 10, Appendix a.

making energy averages in cross sections. For the latter purpose we derive a more suitable expression of W_{ss}, which is shown to be completely equivalent to Eq. (27) in Appendix A.

The level matrix expression for $\mathbf{W}$ is[26] (for positive energy channels)

$$\mathbf{W}=\mathbf{1}+2i\mathbf{P}^{1/2}[\sum_{\lambda\mu}(\gamma_\lambda\times\gamma_\mu)A_{\lambda\mu}]\mathbf{P}^{1/2}, \tag{30a}$$

where

$$(\gamma_\lambda\times\gamma_\mu)_{cc'}\equiv\gamma_{\lambda c}\gamma_{\mu c'}$$

and

$$(\mathbf{A}^{-1})_{\lambda\mu}=(E_\lambda-E_s)\delta_{\lambda\mu}-(\sum_{c\neq t}L_c^0\gamma_{\lambda c}\gamma_{\mu c})-L_t^0\gamma_{\lambda t}\gamma_{\mu t}. \tag{31a}$$

Equation (30a) results from the inversion of the expression $(\mathbf{1}-\mathbf{RL}^0)$ which essentially appears[22] in Eq. (6):

$$(\mathbf{1}-\mathbf{RL}^0)^{-1}=\mathbf{1}+\sum_{\lambda\mu}\gamma_\lambda\times(\mathbf{L}^0\gamma_\mu)A_{\lambda\mu}. \tag{32a}$$

Equivalently one can start with the reduced R matrix expression (15b) and set

$$(\mathbf{1}-\mathfrak{R}\mathbf{L}^0)^{-1}=\mathbf{1}+\sum_{\lambda\mu}\gamma_\lambda^0\times(\mathbf{L}^0\gamma_\mu^0)\mathfrak{A}_{\lambda\mu}. \tag{32b}$$

Equations (30a) and (31a) are then changed to

$$\mathbf{W}=\mathbf{1}+2i\mathbf{P}^{1/2}[\sum_{\lambda\mu}\gamma_\lambda^0\times\gamma_\mu^0\mathfrak{A}_{\lambda\mu}]\mathbf{P}^{1/2}, \tag{30b}$$

$$(\mathfrak{A}^{-1})_{\lambda\mu}=(E_\lambda-E_s)\delta_{\lambda\mu}-L_s^0\gamma_{\lambda s}^0\gamma_{\mu s}^0-L_t^0\gamma_{\lambda t}^0\gamma_{\mu t}^0. \tag{31b}$$

In Eqs. (31) we have separated out the term proportional to L_t^0, since we want to expand $\mathfrak{A}_{\lambda\mu}$ in powers of L_t^0, keeping only the lowest powers. For this purpose we define a matrix $\mathbf{M}$ with components

$$M_{\lambda\mu}=(E_\lambda-E_s)\delta_{\lambda\mu}-L_s^0\gamma_{\lambda s}^0\gamma_{\mu s}^0 \tag{33a}$$

and a matrix $\mathbf{G}$ with components

$$G_{\lambda\mu}=\gamma_{\lambda t}^0\gamma_{\mu t}^0. \tag{34}$$

We wish to mention in passing that if one starts with the form (15a) for the reduced R matrix, the definition of the appropriate $M_{\lambda\mu}$ would change to

$$(E_\lambda+\Delta_{\lambda\lambda}-E_s)\delta_{\lambda\mu}-L_s^0\gamma_{\lambda s}\gamma_{\lambda\mu} \tag{33b}$$

and in Eq. (34) the superscript zero would be left off. With these definitions

$$\mathfrak{A}=\mathbf{M}^{-1}+L_t^0\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1}+(L_t^0)^2\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1}+\cdots, \tag{35}$$

where we show one more power of L_t^0 than we will keep. It follows from Eq. (30b) that to lowest order in L_t^0 or P_t

$$W_{ss}\cong 1+2iP_s\sum_{\lambda\mu}\gamma_{\lambda s}^0\gamma_{\mu s}^0(\mathbf{M}^{-1})_{\lambda\mu}+2iL_t^0P_s\sum_{\lambda\mu}\gamma_{\lambda s}^0\gamma_{\mu s}^0(\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1})_{\lambda\mu}, \tag{36}$$

$$|W_{st}|^2\cong 4P_sP_t|\sum_{\lambda\mu}\gamma_{\lambda s}^0\gamma_{\mu t}^0(\mathbf{M}^{-1})_{\lambda\mu}|^2. \tag{37}$$

As shown in Appendix A, Eq. (36) can be identified term by term with Eq. (21a) and Eq. (37) is equivalent to Eq. (22) so that expression (27) can be obtained once more.

[26] Reference 10, Sec. IX, 1.

V. GENERAL EXPRESSIONS FOR THRESHOLD EFFECT IN CROSS SECTIONS

It is convenient to break up expressions (21a) or (36) into two terms as follows, recalling Eqs. (27) and (28)

$$W_{ss}^0=(1-L_s^{0*}\mathfrak{R}_{ss})/(1-L_s^0\mathfrak{R}_{ss}) \tag{38a}$$

$$=1+2iP_s\sum_{\lambda\mu}\gamma_{\lambda s}^0\gamma_{\mu s}^0(\mathbf{M}^{-1})_{\lambda\mu} \tag{38b}$$

$$=e^{2i(\delta_s-\varphi_s)}, \tag{38c}$$

$$\Delta W_{ss}=2iL_t^0P_s\mathfrak{R}_{st}^2/(1-L_s^0\mathfrak{R}_{ss})^2 \tag{39a}$$

$$=2iL_t^0P_s\sum_{\lambda\mu}\gamma_{\lambda s}^0\gamma_{\mu s}^0(\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1})_{\lambda\mu} \tag{39b}$$

$$=(iL_t^0/P_t^+)\tfrac{1}{2}|W_{st}|^2e^{2i(\delta_s-\varphi_s)}. \tag{39c}$$

If we write

$$U_{ss}=(W_{ss}^0+\Delta W_{ss})e^{2i\varphi_s} \tag{40}$$

and set for either the partial elastic scattering or partial total cross section [Eqs. (1) and (3)]

$$\sigma=\sigma^0+\Delta\sigma, \tag{41}$$

we shall call the threshold effect $\Delta\sigma$ that part of which is proportional to ΔW_{ss} (i.e., L_t^0) and ignore higher powers of ΔW_{ss}. σ^0 will then be that part of the partial cross section which is unaffected by the threshold. It follows immediately from Eqs. (38) to (40) and (1) to (3) that for the partial elastic scattering cross section

$$\sigma_{ss}^0=|1-W_{ss}^0e^{2i\varphi_s}|^2=4\sin^2\delta_s, \tag{42}$$

$$\Delta\sigma_{ss}=-2\,\mathrm{Re}[(1-W_{ss}^{0*}e^{-2i\varphi_s})\Delta W_{ss}e^{2i\varphi_s}] \tag{43a}$$

$$=-\mathrm{Re}[2\Delta W_{ss}e^{2i\varphi_s}-(iL_t^0/P_t^+)\sigma_{st}] \tag{43b}$$

$$=-\mathrm{Re}[(e^{2i\delta_s}-1)(iL_t^0/P_t^+)]\sigma_{st} \tag{43c}$$

$$=-\sigma_{st}\begin{cases}2\sin^2\delta_s.\\ \sin 2\delta_s\end{cases} \tag{43d}$$

The last equation is valid only for $l_t=0$ and no Columb effect in channel t and corresponds to I Eq. (22). It is amusing to note that for $E_s>E_{\mathrm{thr}}$, $\Delta\sigma_{ss}/\sigma_{ss}^0=-\frac{1}{2}\sigma_{st}$. For the partial total cross section one finds

$$\sigma_s^0=2[1-\mathrm{Re}(W_{ss}^0e^{2i\varphi_s})]=4\sin^2\delta_s=\sigma_{ss}^0, \tag{44}$$

$$\Delta\sigma_s=-2\,\mathrm{Re}(\Delta W_{ss}e^{2i\varphi_s}) \tag{45a}$$

$$=-\mathrm{Re}[e^{2i\delta_s}(iL_t^0/P_t^+)]\sigma_{st} \tag{45b}$$

$$=\sigma_{st}\begin{cases}\cos 2\delta_s\\ -\sin 2\delta_s\end{cases}. \tag{45c}$$

The last equation again is valid only for $l_t=0$ and no Coulomb effect in channel t and corresponds to I Eq. (23). The result (44) is obvious since all reaction effects have been eliminated from σ_{ss}^0 and σ_s^0. We should recall that in Eqs. (43) and (45) σ_{st} is meant to be a function of the absolute quantity ρ_t defined in Eq. (9).

VI. THRESHOLD EFFECTS IN ENERGY-AVERAGED CROSS SECTIONS

A. Energy Average with Random Sign Approximation for γ_λ

In I it was stated without general proof that under the assumption of many resonances in the energy-averaging interval one could expect

$$\langle\Delta\sigma_{ss}\rangle_{\rm av}\equiv\begin{cases}-2\langle\sigma_{st}\sin^2\delta_s\rangle_{\rm av}\\ -\langle\sigma_{st}\sin 2\delta_s\rangle_{\rm av}\end{cases}=\begin{cases}-\bar{\sigma}_{st}\\ 0\end{cases},\quad \bar{x}\equiv\langle x\rangle_{\rm av} \tag{46}$$

and

$$\langle\Delta\sigma_s\rangle_{\rm av}\equiv\begin{cases}\langle\sigma_{st}\cos 2\delta_s\rangle_{\rm av}\\ -\langle\sigma_{st}\sin 2\delta_s\rangle_{\rm av}\end{cases}=\begin{cases}0\\ 0\end{cases}, \tag{47}$$

where $\langle\ \rangle_{\rm av}$ implies an energy average over an interval I containing many resonances, and in our notation is equivalent to a bar over the symbol. $\bar{\sigma}_{ss}{}^0$ and $\bar{\sigma}_s{}^0$ would obviously show no threshold effect, since all threshold effects have been eliminated from δ_s by the definition (28). The right-hand sides of Eqs. (46) and (47) were shown in I to follow very simply from the optical-model assumption[2] that the optical-model quantity $\langle U_s\rangle$ should be equal to the energy-averaged $\bar{U}_{ss}$.

From Eqs. (43b) and (45a) it is apparent that if one can prove $\langle\Delta W_{ss}\rangle_{\rm av}=0$, Eqs. (46) and (47) follow immediately. It should be stated here that, as is customary,[8,27] we assume that the channel quantities S, P, and φ are practically constant in the energy interval I over which averages are taken. We shall prove that $\langle\Delta W_{ss}\rangle_{\rm av}=0$ if the signs of $\gamma_{\lambda c}$ are random, by using the following remarks by Thomas.[8] Since all the poles of U_{cc} are situated in the lower half of the complex energy plane, with the exception of those on the real axis associated with bound states, any path of integration involving U_{cc} may be displaced, without crossing poles, upwards in the energy plane, as shown in Fig. 1. According to Thomas,[8] if the averaging interval, called I, contains many resonances it may be presumed that the contributions from the connecting sides of the contour effectively cancel.[28] These statements apply also to W_{cc} or to any part of it, in particular to ΔW_{ss}. Hence we can write, using the symbols shown in Fig. 1,

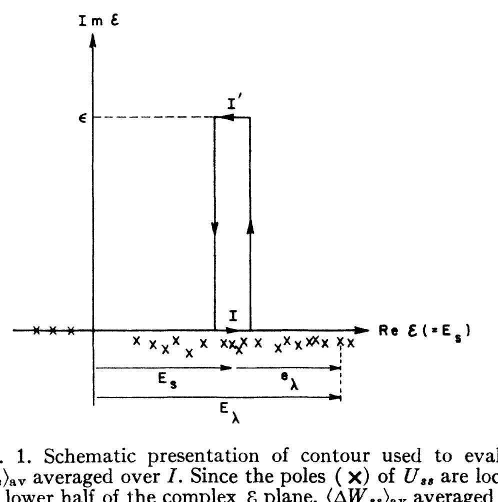

FIG. 1. Schematic presentation of contour used to evaluate $\langle\Delta W_{ss}\rangle_{\rm av}$ averaged over I. Since the poles (×) of U_{ss} are located in the lower half of the complex $\mathcal{E}$ plane, $\langle\Delta W_{ss}\rangle_{\rm av}$ averaged over I will be equal to $\langle\Delta W_{ss}\rangle_{\rm av}$ averaged over I', since the contributions from the sides of the contour cancel. Also indicated on this figure are the symbols used to evaluate expression (51) by means of expression (52). It should be noted that the upper half of the complex energy plane shown is on the first Riemann sheet and the lower half on the second Riemann sheet [J. D. Walecka (private communication)].

$$\langle\Delta W\rangle_{\rm av\ in\ I}=\langle\Delta W_{ss}\rangle_{\rm av\ in\ I'}=\Delta W_{ss}(\mathcal{E}), \tag{48}$$

where

$$\mathcal{E}=E_s+i\epsilon. \tag{49}$$

The last equality in Eq. (48) results from the assumption that if ϵ is sufficiently large, $\Delta W_{ss}(\mathcal{E})$ will be a smooth function of the energy. Using Eqs. (15b) and (39a) we evaluate the expression of ΔW_{ss} in terms of the reduced R matrix (note that $s\neq t$ in the summation):

$$\Delta W_{ss}(\mathcal{E})=2iL_t{}^0P_s\times\frac{\sum_{\lambda\lambda'}\gamma_{\lambda s}{}^0\gamma_{\lambda t}{}^0\gamma_{\lambda' s}{}^0\gamma_{\lambda' t}{}^0/[(E_\lambda-\mathcal{E})(E_{\lambda'}-\mathcal{E})]}{[1-L_s{}^0\mathfrak{R}_{ss}(\mathcal{E})]^2}. \tag{50}$$

Under the assumption of random and uncorrelated signs of $\gamma_{\lambda s}$ and $\gamma_{\lambda t}$ the terms in the numerator with $\lambda\neq\lambda'$ can be expected to cancel out so that the only terms left in the numerator are

$$\sum_\lambda(\gamma_{\lambda s}{}^0)^2(\gamma_{\lambda t}{}^0)^2/(E_\lambda-\mathcal{E})^2. \tag{51}$$

By converting this expression into an integral we can show that it is zero under reasonable conditions. Call $e_\lambda=E_\lambda-E_s$ (see Fig. 1) and let $n(e_\lambda)$ be the density of poles per unit energy range e_λ. Expression (51) then can be converted into an integral, which is zero if $n(e_\lambda)$ does not increase too rapidly as $e_\lambda\rightarrow\infty$ in the complex e_λ plane:

$$\int_{-\infty}^{\infty}\langle(\gamma_{\lambda s}{}^0)^2(\gamma_{\lambda t}{}^0)^2\rangle_{\rm av}n(e_\lambda)de_\lambda/(e_\lambda-i\epsilon)^2=0. \tag{52}$$

As long as $E_s\gg\epsilon$, the extension of the lower limit to $-\infty$ is justified. This evaluation assumes that it is permissible to group close-lying poles in such a way that their average strength $\langle(\gamma_{\lambda s}{}^0)^2(\gamma_{\lambda t}{}^0)^2\rangle_{\rm av}$ is a regular function of the energy e_λ and that their density per unit energy $n(e_\lambda)$ has a meaning. Since the energy ϵ in expression (51) can be made much larger than the level spacing, these assumptions appear to be reasonable.

By a method similar to the one just used one can show that the denominator of the right side of Eq. (50) is not zero and therefore, under the assumption of

[27] Reference 10, Chap. XI.

[28] We are indebted to Dr. P. A. Moldauer for drawing attention to these remarks by Thomas.

random signs of $\gamma_{\lambda s}$, $\gamma_{\lambda t}$

$$\langle \Delta W_{ss} \rangle_{\mathrm{av}} = 0. \tag{53}$$

Although this derivation is based on the special form (15b) of the reduced R matrix, it is equally valid—and more properly executed—for the form (15a), even though the extension of the integral in Eq. (52) to $+\infty$ requires the introduction of an imaginary width term[8] in the denominator of expression (15a). This means that the result (53) really depends on the random sign approximation in two places, once in the form (15a) of the reduced R matrix and once in the evaluation of the numerator of Eq. (50).

The equality (48) depends on the effective cancellation of the path integrals along the sides of the contour shown in Fig. 1. Such a cancellation will occur only if the "slowly varying" quantities, L_t^0 and L_s^0 do not vary appreciably in the interval I. For $l_t = 0$ the derivation is therefore not valid right up to threshold. This point is discussed by Moldauer.[29] It is of interest to note that the proof of Eq. (53) is independent of the width to spacing ratio of the resonances in channels s and t.

Having demonstrated that, under the random sign assumption of the $\gamma_{\lambda c}$, $\langle \Delta W_{ss} \rangle_{\mathrm{av}} = 0$, we see from Eq. (38) that under the same conditions[30]

$$\bar{W}_{ss} = \bar{W}_{ss}^0 \tag{54}$$

and, as mentioned before, Eqs. (43b) and (45a) lead to the "optical-model results" Eqs. (46) and (47), respectively. Indeed, Eq. (54) is essentially the basic assumption of the optical model,[2] that the average of the diagonal component of the collision matrix is a pure entrance channel quantity which shows no discontinuities at any reaction threshold.

For the purposes of Sec. B it is useful to rederive Eq. (53) from the level matrix expression (39b), which, recalling the definition (34), can be written

$$\Delta W_{ss} = 2iL_t^0 P_s \times \sum_{\lambda\lambda'\mu\mu'} \gamma_{\lambda s}^0 \gamma_{\lambda' t}^0 \gamma_{\mu s}^0 \gamma_{\mu' t}^0 (\mathbf{M}^{-1})_{\lambda\lambda'} (\mathbf{M}^{-1})_{\mu'\mu}. \tag{55}$$

The random sign approximation assumes that the signs of two γ quantities for different channels are uncorrelated, but no statement can really be made about the signs of two γ quantities for the same channel. Hence, in taking the average of expression (55) under the random sign approximation we are left with the following terms:

$$\begin{aligned} \langle \Delta W_{ss} \rangle_{\mathrm{av}} = 2iL_t^0 P_s & \times \{ \langle \sum_{\lambda\lambda'\mu'} (\gamma_{\lambda s}^0)^2 \gamma_{\lambda' t}^0 \gamma_{\mu' t}^0 (\mathbf{M}^{-1})_{\lambda\lambda'} (\mathbf{M}^{-1})_{\mu'\lambda} \rangle_{\mathrm{av}} \\ & + \langle \sum_{\lambda\lambda'\mu} \gamma_{\lambda s}^0 \gamma_{\mu s}^0 (\gamma_{\lambda' t}^0)^2 (\mathbf{M}^{-1})_{\lambda\lambda'} (\mathbf{M}^{-1})_{\lambda'\mu} \rangle_{\mathrm{av}} \\ & + \langle \sum_{\lambda\lambda'} \gamma_{\lambda s}^0 (\gamma_{\lambda' s}^0)^2 [(\mathbf{M}^{-1})_{\lambda\lambda'}]^2 \rangle_{\mathrm{av}} \}. \end{aligned} \tag{56}$$

The result that each one of the three average terms on the right side of this expression is equal to zero hinges on these facts:

(1) Any off-diagonal element of $\mathbf{M}^{-1}$ can be represented by a convergent series of terms which are level sums over products of quantities like

$$\gamma_{\lambda s}^0 / [E_\lambda - E_s - L_s^0 (\gamma_{\lambda s}^0)^2] \equiv \gamma_{\lambda s}^0 / \epsilon_\lambda. \tag{57a}$$

The convergence of this series is demonstrated in Appendix B.

(2) If in a series of products of the kind

$$\sum_{\lambda\mu\nu\cdots} F(E_s) / [(\epsilon_\lambda)^l (\epsilon_\mu)^m (\epsilon_\nu)^n \cdots], \tag{57b}$$

where $F(E_s)$ is a regular function of the energy E_s and l, m, n are integers greater or equal to unity, there are ever any multiple poles (i.e., l, m, or n greater than unity), then the average of such a sum is zero in any energy interval which contains many resonances.[31] One should note that the poles of expressions (57) lie in the same (lower) half of the complex energy plane. Also in calculating the average of expression (57b) in any energy interval I one can make the usual assumption that

$$\langle \sum\nolimits_{\lambda\mu\nu} \cdots \rangle_{\mathrm{av\ in\ } I} = (\text{number of terms in } I) \times \int_{-\infty}^{\infty} \cdots dE_s / I. \tag{58}$$

(3) Every term in the three sums in Eq. (56) contains at least one double pole. We illustrate this by noting that, as shown in Appendix B, every term in the (convergent) series expansion of $(\mathbf{M}^{-1})_{\lambda\lambda'} (\mathbf{M}^{-1})_{\mu'\lambda}$ contains the product

$$(L_s^0)^2 (\gamma_{\lambda s}^0)^2 \gamma_{\lambda' s}^0 \gamma_{\mu' s}^0 / [(\epsilon_\lambda)^2 (\epsilon_{\lambda'}) (\epsilon_{\mu'})].$$

[29] P. A. Moldauer, Phys. Rev. (to be published).

[30] Using the above argument of Thomas, which allows replacing $\mathfrak{R}_{ss}$ in Eq. (38) by $\langle \mathfrak{R}_{ss} \rangle$ averaged over I' when calculating $\bar{W}_{ss}^0$ averaged over I, Moldauer has shown that $\bar{W}_{ss}^0 = [1 - (\langle \tau_s \rangle / 4)] / [1 + \langle \tau_s \rangle / 4]$, where $\langle \tau_s \rangle \equiv 4\pi P_s \langle \gamma_{\lambda s}^2 \rangle / D$, the average being taken over the resonances λ and D being the average level distance. P. A. Moldauer (private communication) and reference 29.

[31] The proof of this is easily seen under the Thomas assumptions previously used to arrive at Eq. (52) from the numerator of Eq. (39a). Under those assumptions the average of expression (57b) can be written (at any energy E_s)

$$F(E_s) \int_{-\infty}^{\infty} \frac{n(e_\lambda) de_\lambda}{(e_\lambda - i\epsilon)^l} \int_{-\infty}^{\infty} \frac{n(e_\mu) de_\mu}{(e_\mu - i\epsilon)^m} \int_{-\infty}^{\infty} \frac{n(e_\nu) de_\nu}{(e_\nu - i\epsilon)^n} \cdots,$$

which is zero if any one of the exponent integers l, m, n, $\cdots$ is greater than unity. This method of evaluating expression (57b), with all exponents l, m, n, $\cdots$ equal to unity can be shown to lead to the evaluation of $\bar{W}_{ss}^0$ given in footnote 30, starting from Eqs. (A6) and (B4), provided that $\langle \tau_s \rangle / 4 \ll 1$. [See also sentence preceding Eq. (52)].

This proof of Eq. (53) again assumes that L_t^0 in particular does not vary appreciably in the energy-averaging interval I. If it does, the extension of the integration in Eq. (58) to $\pm\infty$ does not seem justified. It should be noted again that this proof makes no assumption about the width to spacing ratio for the resonances in channels s and t and that the proof could have been done with the form (33b) for the elements of $\mathbf{M}^{-1}$.

B. Partial Breakdown of the Random Sign Approximation

If there is a correlation of the signs of $\gamma_{\lambda c}$, such a correlation could take many different mathematical forms. We wish to examine one form, proposed by Lane and Thomas to take into account direct interaction effects.[11] They assumed that in the neighborhood of any energy E_s the diagonal elements of $\mathbf{M}^{-1}$ give resonances which locally have random signs of $\gamma_{\lambda c}$, but that the off-diagonal elements contribute a slowly varying function of E_s which we call Z. In particular we write the sums occurring in Eqs. (37) and (38b) as follows:

$$\sum_{\lambda\mu} \gamma_{\lambda s}^0 \gamma_{\mu s}^0 (\mathbf{M}^{-1})_{\lambda\mu} = \sum_\lambda (\gamma_{\lambda s}^0)^2 (\mathbf{M}^{-1})_{\lambda\lambda} + Z_{ss}, \quad (59a)$$

$$\sum_{\lambda\mu} \gamma_{\lambda s}^0 \gamma_{\mu t}^0 (\mathbf{M}^{-1})_{\lambda\mu} = \sum_\lambda \gamma_{\lambda s}^0 \gamma_{\lambda t}^0 (\mathbf{M}^{-1})_{\lambda\lambda} + Z_{st}, \quad (59b)$$

and for completeness

$$\sum_{\lambda\mu} \gamma_{\lambda t}^0 \gamma_{\mu t}^0 (\mathbf{M}^{-1})_{\lambda\mu} = \sum_\lambda (\gamma_{\lambda t}^0)^2 (\mathbf{M}^{-1})_{\lambda\lambda} + Z_{tt}, \quad (59c)$$

which would occur in W_{tt}. As Lane and Thomas indicate,[11] these expressions are expected to be valid in any energy interval which is large compared to the widths of compound resonance levels and small compared to the width of single-particle resonances.

To calculate the resulting threshold effect in the partial elastic and total cross sections we consider in turn the following quantities. From Eqs. (38b) and (59a), we obtain by the method indicated in Eq. (58) and using the fact that $(\mathbf{M}^{-1})_{\lambda\lambda} = 1/M_{\lambda\lambda}$ [see Eq. (33a)]

$$\begin{aligned} W_{ss}^0 &= 1 + 2iP_s[i\pi\langle(\gamma_{\lambda s}^0)^2\rangle/D + Z_{ss}] \\ &\equiv 1 - \tfrac{1}{2}\langle\tau_s\rangle + 2iP_s Z_{ss}, \end{aligned} \quad (60)$$

where D is the average level separation and the sign $\langle\ \rangle$ implies an average over resonance levels λ within the energy interval I which is assumed to be very large compared to $P_s(\gamma_{\lambda s}^0)^2$ or D, whichever is larger. The quantities τ have been introduced by Moldauer[30,32]; indeed Eq. (60) corresponds exactly to Eq. (12) in reference 32.

From Eqs. (37) to (39) and (59b) it follows that

$$\Delta W_{ss} = 2iL_t^0 P_s [\sum_\lambda \gamma_{\lambda s}^0 \gamma_{\lambda t}^0 (\mathbf{M}^{-1})_{\lambda\lambda} + Z_{st}]^2,$$

where it should be emphasized that the square brackets do not represent absolute value signs. After averaging the square, term by term, under the same assumptions that have been used to average Eq. (55), we find

$$\langle \Delta W_{ss} \rangle_{\text{av}} = 2iL_t^0 P_s Z_{st}^2. \quad (61)$$

This result depends on the random sign approximation which is expected to be valid in any finite energy region. For a closer discussion of this point we refer to reference 11.

For calculations of the threshold effects in the partial elastic and total cross sections we use expressions (43b) and (45a), respectively, as well as Eq. (61).

$$\begin{aligned} \langle \Delta\sigma_{ss} \rangle_{\text{av}} &= -\operatorname{Re}[(iL_t^0/P_t^+)(4P_t^+ P_s Z_{st}^2 e^{2i\varphi_s} - \bar{\sigma}_{st})] \\ &= \begin{cases} 4P_t^+ P_s |Z_{st}|^2 \cos 2(\varphi_s + \zeta) - \bar{\sigma}_{st} \\ -4P_t^+ P_s |Z_{st}|^2 \sin 2(\varphi_s + \zeta) \end{cases}, \end{aligned} \quad (62)$$

where we have set

$$Z_{st} = |Z_{st}| e^{i\zeta} \quad (63)$$

and assumed $l_t = 0$. Other cases are easily obtained from Eqs. (11) and (7).

$$\begin{aligned} \langle \Delta\sigma_s \rangle_{\text{av}} &= -2 \operatorname{Re}[2i(L_t^0/P_t^+) P_t^+ P_s Z_{st}^2 e^{2i\varphi_s}] \\ &= \begin{cases} 4P_t^+ P_s |Z_{st}|^2 \cos 2(\varphi_s + \zeta) \\ -4P_t^+ P_s |Z_{st}|^2 \sin 2(\varphi_s + \zeta) \end{cases}. \end{aligned} \quad (64)$$

For completeness we also give an expression for $\bar{\sigma}_{st}$ derived under the same assumptions as those leading to Eqs. (60) and (61)

$$\bar{\sigma}_{st} = \langle \tau_t \rangle + 4P_t^+ P_s |Z_{st}|^2. \quad (65)$$

This corresponds to Eq. (38) of reference (32), except that some of the fluctuation and interference terms have been lumped into Z_{st} by means of Eq. (59b). Also terms of second order in $\langle\tau_t\rangle$ have been neglected, as is consistent with the spirit of the present paper.

The important feature of Eqs. (62) and (64) is that they predict a cusp effect in the total cross section, as the result of the assumption of partial breakdown of the random signs of $\gamma_{\lambda c}$. This is contrary to the optical-model assumption that $\bar{W}_{ss}$ and hence the total cross section do not show any discontinuity at thresholds. [Obviously, Eq. (61) does predict a sharp discontinuity if $l_t = 0$.] In principle, it should be possible to decide by experiment, whether Eq. (64) or Eq. (47) is valid in any particular case and so to find out whether the partial breakdown of the random sign assumption is a good physical approximation. In practice, unfortunately the total as well as other cross sections show fluctuations due to the variations in the widths of the resonance levels as well as the approximately random signs of $\gamma_{\lambda c}$ in any small energy interval[33] and the search for a cusp in the total cross section may be difficult. That an expression for the differential elastic cross section analogous to (46) appears to be valid for neutron inter-

[32] P. A. Moldauer, Phys. Rev. 123, 968 (1961). Since in this reference the level shift was set equal to zero for all levels, no superscript zero appears on the $\gamma_{\lambda c}$.

[33] T. Ericson, Phys. Rev. Letters 5, 430 (1960).

actions with Ce^{140} was shown in I, which was based on neutron cross-section measurements of Wells, Tucker, and Meyerhof[34,35] on Ce^{140}.

In I it was assumed that situations may exist in which the phase δ_s of U_{ss} shows fluctuations with energy E_s around a finite mean value $\bar{\delta}_s$ (slowly varying with E_s) in any energy interval containing many resonances:

$$\delta_s=\bar{\delta}_s+\Delta\delta_s,\quad \langle\Delta\delta_s\rangle_{\rm av}=0. \tag{66}$$

We will now show that this assumption is completely equivalent to the aforementioned partial breakdown of the random sign assumption for the $\gamma_{\lambda c}$. On the other hand, in I we showed already that the assumption

$$U_{ss}=\bar{U}_{ss}+\Delta U_{ss},\quad \langle\Delta U_{ss}\rangle_{\rm av}=0 \tag{67}$$

leads to Eqs. (46) and (47) which were derived assuming completely random signs for the $\gamma_{\lambda c}$. It should be noted, in comparing Eqs. (66) and (67), that an energy average of δ_s (or $\Delta\delta_s$) in not equivalent to an energy average of U_{ss} (or ΔU_{ss}).

Although it was not done so in I, we define δ_s by means of Eq. (28), and using Eqs. (27) and (4) write

$$U_{ss}=e^{2i\delta_s}[1-(iL_t^0/P_t^+)\tfrac{1}{2}\sigma_{st}]. \tag{68}$$

Breaking up U_{ss} as in Eq. (40) and using the expressions corresponding to Eqs. (43a) and (45a) we find with the substitution (66)

$$\langle\Delta\sigma_{ss}\rangle_{\rm av}=\begin{cases}\langle\sigma_{st}\cos2\Delta\delta_s\rangle_{\rm av}\cos2\bar{\delta}_s-\bar{\sigma}_{st}\\ -\langle\sigma_{st}\cos2\Delta\delta_s\rangle_{\rm av}\sin2\bar{\delta}_s\end{cases}, \tag{69}$$

$$\langle\Delta\sigma_{s}\rangle_{\rm av}=\begin{cases}\langle\sigma_{st}\cos2\Delta\delta_s\rangle_{\rm av}\cos2\bar{\delta}_s\\ -\langle\sigma_{st}\cos2\Delta\delta_s\rangle_{\rm av}\sin2\bar{\delta}_s\end{cases}, \tag{70}$$

where we have put, as seems reasonable from Eq. (66),

$$\langle\sigma_{st}\sin2\Delta\delta_s\rangle_{\rm av}=0. \tag{71}$$

The equivalence of Eqs. (69) and (70) to Eqs. (62) and (64), respectively, is easily proven by going back to Eq. (39c) and averaging both sides over an energy interval which contains many resonances. On the left side of Eq. (39c) we substitute Eq. (61) and on the right side Eq. (66):

$$|Z_{st}|^2e^{2i\zeta}=[(\langle|W_{st}|^2\cos2\Delta\delta_s\rangle_{\rm av}+i\langle|W_{st}|^2\sin2\Delta\delta_s\rangle_{\rm av})/(4P_t^+P_s)]e^{2i(\bar{\delta}_s-\varphi_s)}.$$

Under the assumption (71) we find the desired result

$$4P_t^+P_s|Z_{st}|^2=\langle|W_{st}|^2\cos2\Delta\delta_s\rangle_{\rm av}=\langle\sigma_{st}\cos2\Delta\delta_s\rangle_{\rm av} \tag{72}$$

and

$$\varphi_s+\zeta=\bar{\delta}_s. \tag{73}$$

[34] J. T. Wells, A. B. Tucker, and W. E. Meyerhof (to be published).

[35] A. B. Tucker, J. T. Wells, and W. E. Meyerhof (to be published).

Although we have just shown that the assumption $\delta_s=\bar{\delta}_s+\Delta\delta_s$ and Eq. (71) are compatible with the previous treatment, we have not yet shown that $\langle\Delta\delta_s\rangle_{\rm av}=0$ is also compatible. This argument is more involved and depends in essence on the unitarity of $\mathbf{W}$. We only sketch the proof. Averaging both sides of Eq. (38c) we obtain with the help of Eq. (60) and substitution (66)

$$1-\tfrac{1}{2}\langle\tau_s\rangle+2iP_sZ_{ss}=[\cos2(\bar{\delta}_s-\varphi_s)+i\sin2(\bar{\delta}_s-\varphi_s)]\times(\langle\cos2\Delta\delta_s\rangle_{\rm av}+i\langle\sin2\Delta\delta_s\rangle_{\rm av}).$$

Equivalently to $\langle\Delta\delta_s\rangle_{\rm av}=0$ we set

$$\langle\sin2\Delta\delta_s\rangle_{\rm av}=0 \tag{74}$$

and obtain with the help of Eq. (73)

$$\tan2\zeta=2P_s\,{\rm Re}Z_{ss}/[1-\tfrac{1}{2}\langle\tau_s\rangle-2P_s\,{\rm Im}Z_{ss}]. \tag{75}$$

We shall prove that this equation follows also from the unitarity of $\mathbf{W}$, and hence that Eq. (74) is indeed compatible with the treatment in the first part of this section.

To shorten the notation we introduce the channel quantity $F_{cc'}$ for Eqs. (59)

$$F_{cc'}=\textstyle\sum_\lambda\gamma_{\lambda c}{}^0\gamma_{\lambda c'}{}^0(\mathbf{M}^{-1})_{\lambda\lambda}+Z_{cc'}. \tag{76}$$

Unitarity of W to first order in L_t^0 or P_t^+ can then be shown to require

$$(1+2iP_sF_{ss})F_{st}^*=F_{st}, \tag{77a}$$

$${\rm Im}F_{ss}=P_s|F_{ss}|^2, \tag{77b}$$

$${\rm Im}F_{tt}=P_s|F_{st}|^2, \tag{77c}$$

$${\rm Re}F_{tt}=0. \tag{77d}$$

Averaging both sides of Eq. (77a) subject to the condition $\langle\gamma_{\lambda s}\gamma_{\lambda t}\rangle=0$ one finds

$$1+2iP_s[(i\pi\langle(\gamma_{\lambda s}{}^0)^2\rangle/D)+Z_{ss}]Z_{st}^*=Z_{st}, \tag{78a}$$

which is just Eq. (75), recalling Eq. (63). Unitarity of $\mathbf{W}$ therefore completes the proof of the complete equivalence of the first part of this section with the assumptions (66).

For completeness we give the results of energy averaging the remaining Eqs. (77):

$${\rm Im}Z_{ss}=[P_s/(1-\tfrac{1}{2}\langle\tau_s\rangle)]\times[|Z_{ss}|^2+(\pi/D)^2\langle(\gamma_{\lambda s}{}^0)^2(\gamma_{\mu s}{}^0)^2\Phi\rangle], \tag{78b}$$

where Φ is the correlation function $\Phi[(\Gamma_\lambda+\Gamma_\mu)/2D]$ introduced by Moldauer,[32]

$${\rm Im}Z_{tt}=P_s|Z_{st}|^2, \tag{78c}$$

$${\rm Re}Z_{tt}=0. \tag{78d}$$

VI. THRESHOLD EFFECT IN THE ONE-LEVEL APPROXIMATION

Under the assumption that a single level is effective in determining the main features of the elastic and total cross sections, we can use the general expressions (43c or d) and (45b or c) for the threshold effects. All that is required in addition to these expressions is the dependence of δ_s on E_s. The reduced R matrix is conveniently used, because Eq. (28) then leads to the desired energy dependence of δ_s. From Eq. (15b) we obtain in the one-level approximation[12]

$$\mathfrak{R}_{ss}=(\gamma_{\lambda s}{}^0)^2/(E_\lambda-E_s)+\mathfrak{R}_{ss}{}^0,$$

where $\mathfrak{R}_{ss}{}^0$ is a slowly varying function of E_s. It is worthy of note that no particular assumption is made about the form of any of the other matrix elements of $\mathfrak{R}^0$.

After a little algebra we obtain from Eq. (28)

$$\tan(\delta_s-\varphi_s)=\frac{P_s(\gamma_{\lambda s}')^2+(E_\lambda-E_s)P_s\mathfrak{R}_{ss}{}^0}{E_\lambda-E_s-S_s{}^0(\gamma_{\lambda s}')^2}, \quad (79)$$

where

$$(\gamma_{\lambda s}')^2=(\gamma_{\lambda s}{}^0)^2/(1-S_s{}^0\mathfrak{R}_{ss}{}^0). \quad (80)$$

Usually it is possible to set $S_s{}^0=0$ or at least to make it very small by appropriate choice of B_s [see Eqs. (7), (11a), and (12a)].

VII. CONCLUSIONS

In this paper we have examined in some detail effects produced in elastic and total cross sections in the neighborhood of a reaction threshold E_{thr}. R-matrix theory allows a precise examination of this problem, as first shown by Wigner.[3] If the phase shift δ_s of that entering partial wave s which gives rise to the reaction is properly defined [see Eq. (28)] there is no need to assume[5–7] that in the expressions for the collision matrix element U_{ss} near E_{thr}, δ_s has to be constant or slowly varying. There may be arbitrarily many, possibly overlapping, resonances near E_{thr} without affecting the validity of the general expressions summarized in Sec. V.

We have shown that the random sign approximation of the $\gamma_{\lambda c}$ leads to the threshold effects derived in I under optical-model assumptions; the equivalent assumption for U_{ss} is $U_{ss}=\bar{U}_{ss}+\Delta U_{ss}$, $\langle\Delta U_{ss}\rangle_{\text{av}}=0$, where $\bar{U}_{ss}$ is slowly varying with E_s. If there is a breakdown in the random sign approximation, a cusp appears in the total cross section, whose presence in principle would signal this breakdown, but in practise might be swamped by the "natural" fluctuations[33] of cross sections. This partial breakdown of the random sign assumption is shown to be equivalent to the assumption $\delta_s=\bar{\delta}_s+\Delta\delta_s$, $\langle\Delta\delta_s\rangle_{\text{av}}=0$, where $\bar{\delta}_s$ is slowly varying with the incident energy.

Lastly, we have given a precise formulation of the threshold effect in a one-level situation.

ACKNOWLEDGMENTS

This work has profited greatly by a series of lectures on R-matrix theory given by P. A. Moldauer at Stanford University. Enlightening discussions with Dr. Moldauer and Professor M. Levy and Professor J. D. Walecka are gratefully acknowledged.

APPENDIX

A. Equivalence of Channel Matrix and Level Matrix Formulation of Threshold Effects

We wish to prove the term by term equivalence of Eqs. (36) and (21a) and Eqs. (37) and (22). We rely heavily on reference 26. Starting with

$$W_{ss}{}^0=1+2iP_s\mathfrak{R}_{ss}/(1-L_s{}^0\mathfrak{R}_{ss}), \quad (A1)$$

we have to obtain $(1-L_s{}^0\mathfrak{R}_{ss})$ by inversion of the matrix[26]

$$(\mathbf{1}-\mathbf{L}^0\boldsymbol{\mathfrak{R}})^{-1}=\mathbf{1}+\sum_{\mu\nu}\gamma_\mu\times(\mathbf{L}^0\gamma_\nu)\mathfrak{A}_{\mu\nu}, \quad (A2)$$

where the matrix $\mathfrak{A}$ is given by Eq. (35). In accordance with Eq. (15b) all γ should carry the superscript zero, but for easier typography we omit this superscript throughout the appendix. Since by definition $W_{ss}{}^0$ does not contain any terms proportional to $L_t{}^0$, such terms occurring in the inversion of Eq. (A2) have to be omitted, yielding

$$1/(1-L_s{}^0\mathfrak{R}_{ss})=1+L_s{}^0\sum_{\mu\nu}\gamma_{\mu s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}, \quad (A3)$$

$$\begin{aligned}W_{ss}{}^0&=1+2iP_s[\textstyle\sum_\lambda\gamma_{\lambda s}{}^2/(E_\lambda-E_s)]\\&\qquad\times[1+L_s{}^0\textstyle\sum_{\mu\nu}\gamma_{\mu s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}]\\&=1+2iP_s[\textstyle\sum_\lambda\gamma_{\lambda s}{}^2/(E_\lambda-E_s)\\&\qquad+L_s{}^0\textstyle\sum_{\lambda\mu\nu}\gamma_{\lambda s}{}^2\gamma_{\mu s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}/(E_\lambda-E_s)]. \quad (A4)\end{aligned}$$

For the sum over ν we use Eq. (1.10)[36] of reference 26 under the consistent neglect of terms proportional to $L_t{}^0$:

$$L_s{}^0\sum_\nu\gamma_{\lambda s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}=(E_\lambda-E_s)(\mathbf{M}^{-1})_{\lambda\mu}-\delta_{\lambda\mu}. \quad (A5)$$

This equation can be obtained also from Eq. (33a). Substitution of Eq. (A5) in Eq. (A4) gives the desired result [see Eq. (38b)]

$$W_{ss}{}^0=1+2iP_s\sum_{\lambda\mu}\gamma_{\lambda s}\gamma_{\mu s}(\mathbf{M}^{-1})_{\lambda\mu}. \quad (A6)$$

For the most convenient formulation ΔW_{ss} we use Eq. (39a)

$$\Delta W_{ss}=2iL_t{}^0P_s\mathfrak{R}_{st}{}^2/(1-L_s{}^0\mathfrak{R}_{ss})^2.$$

With substitution of Eq. (A3)

$$\begin{aligned}\Delta W_{ss}=2iL_t{}^0P_s&[\textstyle\sum_\lambda\gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda-E_s)]\\&\times[\textstyle\sum_{\lambda'}\gamma_{\lambda' s}\gamma_{\lambda' t}/(E_{\lambda'}-E_s)]\\&\times[1+L_s{}^0\textstyle\sum_{\mu\nu}\gamma_{\mu s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}]\\&\times[1+L_s{}^0\textstyle\sum_{\mu'\nu'}\gamma_{\mu' s}\gamma_{\nu' s}(\mathbf{M}^{-1})_{\mu'\nu'}]. \quad (A7)\end{aligned}$$

Combining the first and third bracket and the second and fourth bracket as in Eq. (A4) we obtain with the

[36] In this equation one should replace $\delta_{\lambda\mu}$ by $\delta_{\lambda\nu}$.

help of Eq. (A5)

$$\begin{aligned}\Delta W_{ss}/(2iL_t^0P_s) &= \sum_{\lambda\mu}\gamma_{\lambda t}\gamma_{\mu s}(\mathbf{M}^{-1})_{\lambda\mu}\sum_{\lambda'\mu'}\gamma_{\lambda' t}\gamma_{\mu' s}(\mathbf{M}^{-1})_{\lambda'\mu'}\\ &= \sum_{\lambda\mu\lambda'\mu'}\gamma_{\mu s}\gamma_{\mu' s}(\mathbf{M}^{-1})_{\lambda\mu}\gamma_{\lambda t}\gamma_{\lambda' t}(\mathbf{M}^{-1})_{\lambda'\mu'}\\ &= \sum_{\mu\mu'}\gamma_{\mu s}\gamma_{\mu' s}\sum_{\lambda\lambda'}(\mathbf{M}^{-1})_{\mu\lambda}\gamma_{\lambda t}\gamma_{\lambda' t}(\mathbf{M}^{-1})_{\lambda'\mu'}\\ &= \sum_{\mu\mu'}\gamma_{\mu s}\gamma_{\mu' s}(\mathbf{M}^{-1}\mathbf{G}\mathbf{M}^{-1})_{\mu\mu'}. \qquad \text{(A8)}\end{aligned}$$

The third step makes use of the symmetric nature of $\mathbf{M}^{-1}$. The last line of Eq. (A8) is the third term of Eq. (36) since μ and μ' are arbitrary level indices.

The proof of equivalence of Eqs. (22) and (37) follows similar steps.

$$\begin{aligned}|W_{st}|^2/(4P_sP_t) &= \mathfrak{R}_{st}^2/|1-L_s^0\mathfrak{R}_{ss}|^2\\ &= |[\sum_\lambda \gamma_{\lambda s}\gamma_{\lambda t}/(E_\lambda-E_s)]\\ &\quad\times[1+L_s^0\sum_{\mu\nu}\gamma_{\mu s}\gamma_{\nu s}(\mathbf{M}^{-1})_{\mu\nu}]|^2\\ &= |\sum_{\lambda\mu}\gamma_{\lambda t}\gamma_{\mu s}(\mathbf{M}^{-1})_{\lambda\mu}|^2 \qquad \text{(A9)}\end{aligned}$$

with the help of Eq. (A5). Since $\mathbf{M}^{-1}$ is symmetric, the proof is completed.

B. Convergence of the Expansion of M^{-1}

We shall use the form (33a) for $M_{\lambda\mu}$ and just as in Appendix A leave off the superscript zero on the $\gamma_{\lambda c}$. The proof which follows is easily shown to be equally valid for the form (33b). Following a method suggested by Thomas[8] we expand $(\mathbf{M}^{-1})_{\lambda\mu}$ about the diagonal element $(\mathbf{M}^{-1})_{\lambda\lambda}$ in terms of the off diagonal elements of $\mathbf{M}$. In matrix notation

$$\mathbf{M}^{-1}=\mathbf{m}^{-1}+\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}+\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}+\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}\mathbf{g}\mathbf{m}^{-1}+\cdots, \qquad \text{(B1)}$$

where $\mathbf{m}$ is a diagonal matrix with elements

$$m_{\lambda\lambda}=E_\lambda-E_s-L_s^0\gamma_{\lambda s}^2\equiv\epsilon_\lambda \qquad \text{(B2)}$$

and $\mathbf{g}$ is a pure off-diagonal matrix with elements

$$g_{\lambda\mu}=L_s^0\gamma_{\lambda s}\gamma_{\mu s}. \qquad \text{(B3)}$$

Writing out Eq. (B1)

$$\begin{aligned}(\mathbf{M}^{-1})_{\lambda\mu} &= \delta_{\lambda\mu}/\epsilon_\lambda+[L_s^0\gamma_{\lambda s}\gamma_{\mu s}/(\epsilon_\lambda\epsilon_\mu)]\\ &\quad\times\{1+L_s^0\sum_{\nu\neq\lambda,\mu}\gamma_{\nu s}^2/\epsilon_\nu+(L_s^0)^2(\sum_{\nu\neq\lambda,\mu}\gamma_{\nu s}^2/\epsilon_\nu)\\ &\quad\times(\sum_{\nu'\neq\lambda,\mu,\nu}\gamma_{\nu' s}^2/\epsilon_{\nu'})\cdots+\}. \qquad \text{(B4)}\end{aligned}$$

To estimate the convergence of the series in curly brackets we consider the average value of each sum. We recall the well-known fact,[8,32] following from Eq. (58), that

$$\langle\sum_\lambda \gamma_{\lambda s}^2/\epsilon_\lambda\rangle_{\text{av}}=i\pi\langle\gamma_{\lambda s}^2\rangle/D, \qquad \text{(B5a)}$$

where D is the average level distance between the resonances λ. Now in the sum $\sum_{\nu\neq\lambda,\mu}$ the effective energy level spacing is larger than in the sum $\sum_\lambda$ because certain resonances are omitted from the sum. In the sum $\sum_{\nu'\neq\lambda,\mu,\nu}$ still more resonances are omitted so that we can write, noting that λ, μ, ν, ν', are just running indices,

$$\langle\sum_{\nu\neq\lambda,\mu}\gamma_{\nu s}^2/\epsilon_\nu\rangle_{\text{av}}=i\pi\langle\gamma_{\lambda s}^2\rangle/D_1, \qquad \text{(B5b)}$$

$$\langle\sum_{\nu'\neq\lambda,\mu,\nu}\gamma_{\nu' s}^2/\epsilon_{\nu'}\rangle_{\text{av}}=i\pi\langle\gamma_{\lambda s}^2\rangle/D_2, \text{ etc.} \qquad \text{(B5c)}$$

where $D<D_1<D_2\cdots$. The series of terms in the curly brackets of Eq. (B4) can therefore be estimated to be equal to (setting $S_s^0=0$)

$$\{1-\pi\langle\Gamma_s\rangle/(2D_1)+[\pi\langle\Gamma_s\rangle/(2D_1)][\pi\langle\Gamma_s\rangle/(2D_2)]-\cdots\},$$

where we have represented the average partial level width $\Gamma_{\lambda s}$ by $\langle\Gamma_s\rangle=2P_s\langle\gamma_{\lambda s}^2\rangle$. The absolute ratio of the $(n+1)$st term to the (n)th term will be equal to $\pi\langle\Gamma_s\rangle/(2D_n)$, a quantity which must become less than unity for sufficiently large n, when sufficiently many resonances have been omitted from the level sums occurring in the curly bracket of Eq. (B4). This proves that the series occurring in Eq. (B4) and hence in Eq. (B1) is convergent independent of the width to spacing ratio $\langle\Gamma_s\rangle/D$ in channel s.

It is also obvious from the form of Eq. (B4) that every term in the (convergent) series expansion of the product $(M^{-1})_{\lambda\lambda'}(M^{-1})_{\mu'\lambda}$ will be proportional to $(\epsilon_\lambda)^{-2}$ and hence contain the double pole which was necessary to show that the right side of Eq. (56) is equal to zero.